THE DOWNWASH OF THE FLOW BEHIND THE SWEPT VORTEX OF FINITE SPAN FOR NON-LAMINAR MOTION

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Belotserkovskii in his paper [1] has worked out the computation of the downwash behind a straight vortex of finite span for the case of non-laminar motion.

In this article are presented the formulas which allow to construct a method of calculation of the downwash in an ideal incompressible fluid behind a wing of large aspect ratio and small angle of sweep for the case when the circulation, variable along the span of the wing, also varies harmonically with time.

We shall calculate the downwash of the flow behind the vortex of finite span with variable intensity and with a small angle of sweep, which is immersed in the subsonic flow of an ideal fluid. The downwash velocities are made up of the velocities induced by the quasi-stationary vortex system and by the non-stationary vortex sheet. To the former belong the free vortices (inducing V_{y1}), namely, the vortices parallel to the bound vortices, and the sheet of trailing vortices (inducing V_{y2}) which extend downstream from the bound vortices parallel to the x-axis. These are represented in Fig. 1 by solid lines. To the latter vortex sheet belongs

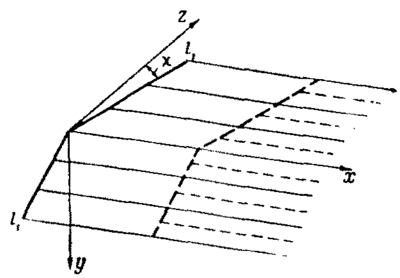

Fig. 1.

the sheet of free non-stationary vortices (inducing W_{y1}) downstream of the bound vortex, to which they are parallel and further the sheet of non-stationary trailing vortices (inducing W_{y2}) which extend downstream from the former and run parallel to the x-axis. The non-stationary vortex system is indicated in dashed lines in Fig. 1.

If the circulation Γ at the bound vortex is given an increment, then according to the theorem of Helmholtz the same increment in magnitude, but opposite in sign, will occur in the circulation in the non-stationary vortex system, emanating from the moving vortex. Accordingly the intensity of the circulation of the non-stationary vortex sheet, for instance, at the point $(x, 0, 0)$ at time t will be

$$\frac{d\Gamma(x, t)}{dt} = -\frac{1}{V_0}\frac{d}{dt}\Gamma\left(t - \frac{x}{V}\right) \tag{1}$$

Let us assume that the circulation of the moving vortex varies with time:

$$\Gamma(z, t) = \Gamma_0(z) f(t) \qquad \left(\frac{d\Gamma_0}{dz} < 0\right) \tag{2}$$

Then according to the theorem of Biot-Savart at the point (x_0, y_0, z_0) the velocity of downwash is

$$V_{y1} = \frac{f(t)}{4\pi}\int_{-l}^{+l}\Gamma_0(z)\frac{x_0 - z\,\mathrm{tg}\,\varkappa}{[(|z|\,\mathrm{tg}\,\chi - x_0)^2 + y_0^2 + (z - z_0)^2]^{3/2}}\,dz \tag{3}$$

$$V_{y2} = -\frac{f(t)}{4\pi}\int_{-l}^{+l}\int_{|z|\,\mathrm{tg}\,\chi}^{\infty}\frac{d\Gamma_0(z)}{dz}\frac{z - z_0}{[(x - x_0)^2 + y_0^2 + (z - z_0)^2]^{3/2}} \tag{4}$$

$$W_{y1} = -\frac{1}{4\pi V_0}\int_{-l}^{+l}\int_{0}^{\infty}\Gamma_0(z)\frac{(x_0 - x^* - |z|\,\mathrm{tg}\,\chi)\,dz\,dx^*}{[(x_0 - x^* - |z|\,\mathrm{tg}\,\chi)^2 + y_0^2 + (z - z_0)^2]^{3/2}}\frac{df(t - x^*/V)}{dt} \tag{5}$$

$$W_{y2} = \frac{1}{4\pi V_0}\int_{0}^{\infty}\frac{df(t - x^*/V)}{dt}\,dx^*\int_{x^* + |z|\,\mathrm{tg}\,\chi}^{\infty}\frac{z\,dx}{[(x - x_0)^2 + y_0^2 + (z - z_0)^2]^{3/2}}\int_{-l}^{+l}\frac{d\Gamma_0(z)}{dz}\,dz \tag{6}$$

where x^* is a running coordinate along the axis of symmetry of the bound vortex. There is a minus sign in front of the integral (4) because the intensity of the connected vortices is negative. In the last integral $d\Gamma/dz > 0$, because this is the derivative of the sheet of the vortices which emanate from the non-stationary free vortices, the direction of the circulation of which is opposite to that of the bound vortex. We shall consider the harmonic case when Γ varies with time:

$$\Gamma = \Gamma_0(z)\,e^{i\omega t} \tag{7}$$

and we shall investigate the downwash of the flow at the axis of the symmetry of the bound vortex. The downwash of the flow from the quasi-stationary shroud will be in phase with the circulation of the bound vortex, whereas the downwash of the flow from the non-stationary vortex sheet lags behind with respect to Γ. Therefore, the downwash will be expressed in the following general form:

$$V_y = V_{y0} e^{i(\omega t - \vartheta)} \tag{8}$$

where θ is the phase lag of the downwash. For further investigation of the integrals we will make use of some interpolation formulas. We shall introduce the angle θ assuming that

$$z = l_1 \cos\theta \quad z_\nu = l_1 \cos\theta_\nu, \quad \theta_\nu = \nu \frac{\pi}{m+1} \quad (\nu = 1, \ldots, m) \tag{9}$$

where m are some fixed, but arbitrary chosen positive numbers.

If $F(\theta)$ is the function which is approximated by the first m members of the Fourier series, then according to the interpolation formula

$$F(\theta) = \sum_{n=1}^{m} F_n R_n \quad \left(R_n = \frac{2}{m+1} \sum_{\tau=1}^{m} \sin\tau\theta_n \sin\tau\theta\right) \tag{10}$$

Upon the computation of the integrals making use of the interpolation formulas we obtain the expressions for the amplitude of the downwash velocity of the flow and for the lag of the downwash:

$$V_{y0} = \frac{1}{4\pi}\left[S_1^2 + (S_2 - S_3)^2\right]^{1/2}, \qquad \vartheta = \operatorname{arctg}\frac{S_1}{S_2 - S_3} \tag{11}$$

$$S_1 = k\sum_{n=1}^{m} \Gamma_n I_n^*, \qquad S_2 = \sum_{n=1}^{m} \Gamma_n I_n, \qquad S_3 = k\sum_{n=1}^{m} \Gamma_n I_n' \quad \left(k = \frac{\omega l_1}{V_0} = \frac{\omega l}{V_0}\frac{1}{\cos\chi}\right) \tag{12}$$

$$I_n^* = \frac{2}{m+1}\left[\sum_{\tau=1}^{m} \sin\tau\theta_n \int_0^\pi \sin\tau\theta \sin\theta \, \frac{\xi_0 - |\cos\theta|\,\mathrm{tg}\,\chi}{\left[(\xi_0 - |\cos\theta|\,\mathrm{tg}\,\chi)^2 + \cos^2\theta\right]^{3/2}}\, d\theta + \right. \tag{13}$$

$$+ \sum_{\tau=1}^{m} \tau \sin\tau\theta_n \int_0^\pi \frac{\cos\tau\theta}{\cos\theta}\left[\frac{|\cos\theta|\,\mathrm{tg}\,\chi - \xi_0}{\left[(\xi_0 - |\cos\theta|\,\mathrm{tg}\,\chi)^2 + \cos^2\theta\right]^{1/2}} + 1\right] d\theta -$$

$$\left. - 2\pi \sum_{\tau=1}^{[\varkappa]} (-1)^\tau (2\tau - 1)\sin(2\tau - 1)\theta_n\right] \qquad \left(\xi_0 = \frac{x_0}{l_1}, \; \varkappa = \mathrm{ent}\,\frac{m+1}{2}\right) \tag{14}$$

$$I_n = \frac{2}{m+1}\sum_{\tau=1}^{m} \sin\tau\theta_n \left[\int_0^\pi \sin\tau\theta \sin\theta \;\; T_1(\theta)\, d\theta + \tau \int_0^\pi \cos\tau\theta \;\; T_2(\theta)\, d\theta\right] \tag{15}$$

$$I_n' = \frac{2}{m+1}\sum_{\tau=1}^{m} \sin\tau\theta_n \left[\int_0^\pi \sin\tau\theta \sin\theta \;\; T_1'(\theta)\, d\theta + \tau \int_0^\pi \cos\tau\theta \;\; T_2'(\theta)\, d\theta\right] \tag{16}$$

$$T_1(\theta) = j_1 \sin(k, a) - j_2 \cos(k, a), \qquad j_1 = \int_{-a}^{\infty} \frac{\gamma}{\delta^3} \sin k\gamma \, d\gamma \tag{17}$$

$$T_2(\theta) = -j_3 \sin(k, a) + j_4 \cos(k, a), \qquad j_2 = \int_{-a}^{\infty} \frac{\gamma}{\delta^3} \cos k\gamma \, d\gamma \tag{18}$$

$$T_1'(\theta) = -j_2 \sin(k, a) - j_1 \cos(k, a), \qquad j_3 = \int_{-a}^{\infty} \left(1 - \frac{\gamma}{\delta}\right) \frac{\sin k\gamma}{\cos\theta}\, d\gamma \tag{19}$$

$$T_2'(\theta) = j_4 \sin(k, a) + j_3 \cos(k, a), \qquad j_4 = \int_{-a}^{\infty} \left(1 - \frac{\gamma}{\delta}\right) \frac{\cos k\gamma}{\cos\theta}\, d\gamma \tag{20}$$

where

$$a = \xi_0 - |\cos\theta| \operatorname{tg}\chi, \qquad \delta^{*} = (\gamma^2 + \cos^2\theta)^{1/2} \tag{21}$$

BIBLIOGRAPHY

1. Belotserkovskii, S.M., Podkovoobrazuyi vikhr' pri neustanevivshemsia dvizhenii (Horseshoe-shaped vortex in the case of non-stationary motion). *PMM* Vol. 19, No. 2, 1955.